

## Exact solution of a $Z(4)$ gauge Potts model on planar lattices

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The exact solution of a general  $Z(4)$  gauge Potts model with a single and double plaquette representation of the action is found on a subspace of gauge-coupling parameters on the square and triangular lattices. The two Ising-type critical lines of a second-order phase transition for the model on a square lattice are found. For the model on a triangular lattice the two critical surfaces of an Ising-type and two nontrivial lines of a second-order phase transition with different critical behavior than on the critical surfaces are found. It is shown that a two-dimensional (2D) general  $Z(4)$  gauge Potts model with single and double plaquette representation of the action and a 2D spin- $\frac{3}{2}$  Ising model belong to the same universality class. [S1063-651X(99)03910-0]

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### I. INTRODUCTION

Confinement-deconfinement phase transition in gauge theory is one of the challenging problems in modern physics. Lattice-gauge theory is a gauge-invariant nonperturbative tool of regularization of gauge action to avoid the ultraviolet divergence in the theory. Although this regularization procedure is not unique, different ways of defining gauge theories on the lattice should lead to the same physics in each case when the continuum limit of vanishing lattice spacing is taken. From a theoretical point of view, investigations with different lattice actions will enable a deeper understanding of the physics of confinement and other related problems in QCD. The first lattice-gauge model with a single plaquette representation of the action has been introduced by Wilson,

$$S = -\beta \sum_p \text{Re } U_p,$$

where  $U_p$  denotes the usual plaquette variable, the product of link gauge fields around a plaquette [1]. It was expected that non-Abelian gauge theories, in general, do not have any phase transitions separating strong- and weak-coupling regimes. Therefore, confinement, explicitly shown on the lattice in the strong-coupling region, should persist also in the continuum limit. Later, Bhanot and Creutz [2] extended the form of the Wilson action by adding an adjoint coupling term. Using Monte Carlo simulations it was shown that confinement could survive even through the phase diagram of the mixed action and the so-called bulk (volume) phase transitions separating strong- and weak-coupling regions exist [3,4].

The choice of action is still far from unique. Recently, several improved actions have been proposed as a way of reducing scaling violation in the approach to the continuum limit from a lattice action. Among them the Symanzik-Wiesner action constructed from a combination of  $(1 \times 1)$  and  $(1 \times 2)$  Wilson loops, the Bhanot-Creutz action [2], the tadpole-improved actions [5], and the  $q$ -state gauge Potts model with a single and double plaquette form of action [6].

The Monte Carlo analysis of the  $q$ -state Potts model with a single and double plaquette form of action [6] showed that for  $d=3$ ,  $q=2$ , first- and second-order transition lines; and for  $d=2$ ,  $q=3,4$ , the second-order; and for  $q=5$ , first-order transitions are existed, which is in good agreement with the analytical results for  $d=2$ .

The lattice-gauge models with double plaquette interaction terms in the action were proposed and studied in three dimensions and four dimensions by Edgar [7] and Bhanot *et al.* [8]. Turban investigated the two-dimensional (2D) gauge model with the global  $Z(2)$  symmetry on a rectangular lattice [9]. He reduced it to the usual spin- $\frac{1}{2}$  Ising model on a square lattice and obtained a point of a second-order phase transition. The  $Z(3)$  gauge model on the flat triangular and square lattices with double plaquette representation of the action was investigated by Ananikian and Shcherbakov [10]. It was reduced to the spin-1 Blume-Emery-Griffiths (BEG) model [11] and an Ising-type critical line of a second-order phase transition was found on a subspace of the interaction constants [12].

The fact that lattice-gauge theories could be mapped to the classical spin systems is well known. For example, Wilczek and Rajagopal [13] showed that in QCD with two flavors of massless quarks, the chiral phase transition is in the same universality class as the classical  $O(4)$  Heisenberg antiferromagnet and they also established a dictionary between QCD and the magnetic system (see also Ref. [14]). Okawa [15], using Monte Carlo renormalization-group methods,

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showed that a (3+1)-dimensional SU(2) lattice-gauge theory and a three-dimensional Ising model belong to the same universality class.

In this paper we consider the generalized Z(4) gauge Potts model with a single and double plaquette representation of the action on the square and triangular lattices. We found an exact analytical solution of this model on a subspace of gauge-coupling parameters. Using duality transformation [16] and exact results for the spin- $\frac{3}{2}$  Ising model on the square and honeycomb lattices [17,18], we investigated the critical properties of the gauge theory. We showed that a two-dimensional generalized Z(4) gauge Potts model with single and double plaquette representation of the action and a two-dimensional spin- $\frac{3}{2}$  Ising model belong to the same universality class.

The paper is organized as follows. In Sec. II we define the model under consideration and present obtained results. Section III contains some concluding remarks.

## II. THE Z(4) GAUGE POTTS MODEL

The most general form of the Z(4) gauge Potts model with a single and double plaquette representation of the action is defined through the action

$$S_{Gauge} = - \sum_{\langle p_i p_j \rangle} \sum_{y,z} \tilde{\beta}_{yz} \delta_{U_{p_i,y}} \delta_{U_{p_j,z}} - \sum_{p_i} \sum_z \tilde{\beta}_z \delta_{U_{p_i,z}}, \quad (1)$$

where the outer summation in the first term runs over all nearest-neighbor plaquettes and in the second one is over all plaquettes of the lattice. The indexes  $y$  and  $z$  of the inner summations run over the group Z(4). The  $U_p = \prod_{b \in \partial p} U_b$  denotes the ordered product of link gauge fields  $U_b$ 's around an elementary plaquette. Each link variable  $U_b$  takes the value  $\exp(ik\pi/2) \in Z(4)$ ,  $k=0,1,2,3$ .  $\delta$  is the standard Kronecker symbol and  $\tilde{\beta}_{yz}, \tilde{\beta}_z$ ,  $y, z \in Z(4)$  are coupling parameters.

From the obvious identity for the Kronecker symbols

$$\delta_{U_{p_i,1}} + \delta_{U_{p_i,z_1}} + \delta_{U_{p_i,z_2}} + \delta_{U_{p_i,z_3}} = 1,$$

where  $z_k \in Z(4)$ ,  $k=1,2,3$ , and an assumption that the coupling parameters are symmetric under the transposition of the indexes  $y$  and  $z$ , we can reduce the number of independent coupling parameters and rewrite gauge action (1) in the following form:

$$\begin{aligned} S_{Gauge} = & - \sum_{\langle p_i p_j \rangle} [\beta_{11} \delta_{U_{p_i,1}} \delta_{U_{p_j,1}} + \beta_{22} \delta_{U_{p_i,z_1}} \delta_{U_{p_j,z_1}} \\ & + \beta_{33} \delta_{U_{p_i,z_2}} \delta_{U_{p_j,z_2}} + \beta_{12} (\delta_{U_{p_i,1}} \delta_{U_{p_j,z_1}} \\ & + \delta_{U_{p_i,z_1}} \delta_{U_{p_j,1}}) + \beta_{13} (\delta_{U_{p_i,1}} \delta_{U_{p_j,z_2}} + \delta_{U_{p_i,z_2}} \delta_{U_{p_j,1}}) \\ & + \beta_{23} (\delta_{U_{p_i,z_1}} \delta_{U_{p_j,z_2}} + \delta_{U_{p_i,z_2}} \delta_{U_{p_j,z_1}})] \\ & + \sum_{p_i} (\beta_1 \delta_{U_{p_i,1}} + \beta_2 \delta_{U_{p_i,z_1}} + \beta_3 \delta_{U_{p_i,z_2}}). \end{aligned} \quad (2)$$

The partition function for this gauge model is defined as a sum of Boltzmann weights  $\exp(S_{Gauge})$  over all configurations of the gauge variables  $\{U\}$ ,

$$Z_{Gauge} = \sum_{\{U\}} \exp[-S_{Gauge}]. \quad (3)$$

To establish the connection between this gauge model and the spin- $\frac{3}{2}$  Ising model, we introduce spin variables  $S_i$  in the sites of the dual lattice such that

$$\begin{aligned} S_i &= \frac{1}{2} (\delta_{U_{p_i,z_0}} - \delta_{U_{p_i,z_1}}) + \frac{3}{2} (\delta_{U_{p_i,z_2}} - \delta_{U_{p_i,z_3}}), \\ S_i^2 &= \frac{1}{4} (\delta_{U_{p_i,z_0}} + \delta_{U_{p_i,z_1}}) + \frac{9}{4} (\delta_{U_{p_i,z_2}} + \delta_{U_{p_i,z_3}}), \\ S_i^3 &= \frac{1}{8} (\delta_{U_{p_i,z_0}} - \delta_{U_{p_i,z_1}}) + \frac{27}{8} (\delta_{U_{p_i,z_2}} - \delta_{U_{p_i,z_3}}). \end{aligned} \quad (4)$$

Using this substitution, we can rewrite gauge action (2) in terms of new spin variables

$$\begin{aligned} S_{Spin} = & - \sum_{\langle ij \rangle} \left[ J S_i S_j + K S_i^2 S_j^2 + L S_i^3 S_j^3 + \frac{M}{2} (S_i S_j^3 + S_i^3 S_j) \right. \\ & \left. + \frac{M_1}{2} (S_i S_j^2 + S_i^2 S_j) + \frac{M_2}{2} (S_i^2 S_j^3 + S_i^3 S_j^2) \right] \\ & - \sum_i (h S_i - \Delta S_i^2 + h_3 S_i^3), \end{aligned} \quad (5)$$

where constants  $J, K, L, M, M_1, M_2, h, \Delta$ , and  $h_3$  are linear combinations of the original gauge-coupling parameters:

$$J = \frac{81}{64} \beta_{11} - \frac{81}{32} \beta_{12} - \frac{3}{32} \beta_{13} + \frac{81}{64} \beta_{22} + \frac{3}{32} \beta_{23} + \frac{1}{576} \beta_{33},$$

$$K = \frac{1}{16} \beta_{11} + \frac{1}{8} \beta_{12} - \frac{1}{8} \beta_{13} + \frac{1}{16} \beta_{22} - \frac{1}{8} \beta_{23} + \frac{1}{16} \beta_{33},$$

$$L = \frac{1}{4} \beta_{11} - \frac{1}{2} \beta_{12} - \frac{1}{6} \beta_{13} + \frac{1}{4} \beta_{22} + \frac{1}{6} \beta_{23} + \frac{1}{36} \beta_{33},$$

$$M = -\frac{9}{8} \beta_{11} + \frac{9}{4} \beta_{12} + \frac{5}{12} \beta_{13} - \frac{9}{8} \beta_{22} - \frac{5}{12} \beta_{23} - \frac{1}{72} \beta_{33},$$

$$M_1 = -\frac{9}{16} \beta_{11} + \frac{7}{12} \beta_{13} + \frac{9}{16} \beta_{22} - \frac{13}{24} \beta_{23} - \frac{1}{48} \beta_{33},$$

$$M_2 = \frac{1}{4} \beta_{11} - \frac{1}{3} \beta_{13} - \frac{1}{4} \beta_{22} + \frac{1}{6} \beta_{23} + \frac{1}{12} \beta_{33},$$

$$\begin{aligned} h = & \gamma \left( \frac{81}{128} \beta_{11} - \frac{3}{32} \beta_{13} - \frac{81}{128} \beta_{22} + \frac{3}{64} \beta_{23} + \frac{1}{384} \beta_{33} \right) + \frac{9}{8} \beta_1 - \frac{9}{8} \beta_2 \\ & - \frac{1}{24} \beta_3, \end{aligned}$$

$$\begin{aligned} \Delta = & \gamma \left( \frac{9}{64} \beta_{11} + \frac{9}{32} \beta_{12} - \frac{5}{32} \beta_{13} + \frac{9}{64} \beta_{22} - \frac{5}{32} \beta_{23} + \frac{1}{64} \beta_{33} \right) + \frac{1}{4} \beta_1 \\ & + \frac{1}{4} \beta_2 - \frac{1}{4} \beta_3, \end{aligned}$$

$$\begin{aligned} h_3 = & \gamma \left( -\frac{9}{32} \beta_{11} + \frac{1}{8} \beta_{13} + \frac{9}{32} \beta_{22} + \frac{1}{16} \beta_{23} - \frac{1}{96} \beta_{33} \right) - \frac{1}{2} \beta_1 + \frac{1}{2} \beta_2 \\ & + \frac{1}{6} \beta_3, \end{aligned}$$

where  $\gamma$  is the coordination number of the dual lattice. Action (5) coincides with the Hamiltonian multiplied by  $1/k_B T$  of the spin- $\frac{3}{2}$  Ising model [17,18]. Thus, the partition function of generalized gauge Potts model (3) is equal up to the factor to the partition function of the spin- $\frac{3}{2}$  Ising model defined on the dual lattice

$$Z_{Gauge} = 4^{a(\gamma)N} Z_{Spin}^{Dual}, \quad (6)$$

where

$$Z_{Spin}^{Dual} = \sum_{\{S\}} \exp[-S_{Spin}],$$

and  $a(\gamma)$  is a constant that depends on the coordination number of the lattice,  $a=1$  for the square lattice, and  $a=\frac{1}{2}$  for the honeycomb one respectively.

A factor  $4^{a(\gamma)N}$  has been included in Eq. (6) to take into account the difference between the number of gauge  $\{U\}$  and spin  $\{S\}$  configurations. To obtain the phase structure of this gauge model we will restrict ourselves to the spin- $\frac{3}{2}$  Ising model on the square and honeycomb lattices. For coincidence with the spin- $\frac{3}{2}$  Ising model the coefficients  $h$ ,  $h_3$ ,  $M_1$ , and  $M_2$  will be set to zero. Thus, we can express the parameters of the spin- $\frac{3}{2}$  model through the rest of the gauge couplings as follows:

$$J = \frac{337}{288} \beta_{11} - \frac{81}{32} \beta_{12} + \frac{1}{288} \beta_{13} + \frac{49}{36} \beta_{22},$$

$$K = \frac{1}{8} \beta_{11} + \frac{1}{8} \beta_{12} - \frac{1}{8} \beta_{13},$$

$$L = \frac{1}{18} \beta_{11} - \frac{1}{2} \beta_{12} + \frac{1}{18} \beta_{13} + \frac{4}{9} \beta_{22},$$

$$M = -\frac{25}{36} \beta_{11} + \frac{9}{4} \beta_{12} - \frac{1}{36} \beta_{13} - \frac{14}{9} \beta_{22},$$

$$\Delta = \gamma \left( \frac{9}{32} \beta_{11} + \frac{9}{32} \beta_{12} - \frac{1}{32} \beta_{13} \right) + \frac{1}{2} \beta_1.$$

The general spin- $\frac{3}{2}$  Ising model on a honeycomb lattice was investigated by Izmailian and Ananikian [18]. The model is described by the Hamiltonian

$$-\beta H = \sum_{\langle ij \rangle} \left[ JS_i S_j + KS_i^2 S_j^2 + LS_i^3 S_j^3 + \frac{M}{2} (S_i S_j^3 + S_i^3 S_j) \right] - \Delta \sum_i S_i^2,$$

where  $\beta = 1/k_B T$ ,  $S_i = \pm \frac{1}{2}$ ,  $\pm \frac{3}{2}$  is the spin variable at site  $i$ , and  $\langle ij \rangle$  indicates the summation over all nearest-neighbor pairs of sites. Under the conditions

$$\tanh^2(J_1) = \tanh(J_2) \tanh(J_0),$$

$$\exp(-4K) = \cosh(J_2 - J_0), \quad (7)$$

that in terms of gauge-coupling parameters are

$$J_0 = \frac{1}{4} \beta_{11} - \frac{1}{2} \beta_{12} + \frac{1}{4} \beta_{22},$$

$$J_1 = \frac{1}{4} \beta_{11} - \frac{1}{4} \beta_{22}, \quad (8)$$

$$J_2 = -\frac{1}{4} \beta_{11} + \frac{1}{2} \beta_{13} + \frac{1}{4} \beta_{22},$$

the model transforms to the spin- $\frac{1}{2}$  Ising model on the same lattice. The free energy and critical point for the spin- $\frac{1}{2}$  Ising model on the honeycomb lattice in the limit of an infinite lattice are well known [19]. Thus, using this result one can obtain the important thermodynamic properties of the spin- $\frac{3}{2}$  Ising model with Z(2) symmetry [17]. After substitution of  $J$ ,  $K$ , and  $L$  from Eq. (8) into Eq. (7) we obtain the subspace in which the corresponding spin- $\frac{3}{2}$  Ising model can be solved exactly,

$$2 - \exp(\beta_{12}) - \exp(\beta_{11} - \beta_{13}) = 0,$$

$$\begin{aligned} & \cosh \frac{1}{2}(\beta_{11} - \beta_{22}) \cosh \frac{1}{2}(\beta_{11} - \beta_{13} - \beta_{12}) \\ & = \cosh \frac{1}{2}(\beta_{22} + \beta_{13} - \beta_{12}). \end{aligned} \quad (9)$$

Then, using the exact solution [18], we obtain the  $\lambda$  surface of an Ising-type transition (logarithmic specific heat singularity) for our Z(4) gauge model

$$\frac{\tanh \frac{1}{4}(\beta_{22} - 2\beta_{12} + \beta_{11}) + \tanh \frac{1}{4}(\beta_{22} + 2\beta_{13} - \beta_{11}) \exp(-2\Delta_0)}{1 + \exp(-2\Delta_0)} = \frac{1}{\sqrt{3}}, \quad (10)$$

where

$$\Delta_0 = \frac{3}{32} (9\beta_{11} + 9\beta_{12} - \beta_{13}) + \frac{1}{2} \beta_1 - 3R,$$

$$\begin{aligned} \exp(-4R) &= \frac{\cosh \frac{1}{4}(\beta_{22} - 2\beta_{12} + \beta_{11})}{\cosh \frac{1}{4}(\beta_{22} + 2\beta_{13} - \beta_{11})} \\ &\quad \times \cosh^{5/4} \frac{1}{2}(\beta_{12} + \beta_{13} - \beta_{11}), \end{aligned}$$

in the space spanned by  $\beta_{12}$ ,  $\beta_{13}$ , and  $\beta_1$ . It is easy to see that the  $\lambda$  surface of the critical points in Eq. (10) is defined only in the two regions of the  $(\beta_{12}, \beta_{13})$  plane,

$$(i) \quad 0 \leq \tanh \frac{1}{4}(\beta_{22} - 2\beta_{12} + \beta_{11}) \leq 1/\sqrt{3} \quad \text{and} \quad 1/\sqrt{3} \leq \tanh \frac{1}{4}(\beta_{22} + 2\beta_{13} - \beta_{11}) \leq 1,$$

$$(ii) \quad 0 \leq \tanh \frac{1}{4}(\beta_{22} + 2\beta_{13} - \beta_{11}) \leq 1/\sqrt{3} \quad \text{and} \quad 1/\sqrt{3} \leq \tanh \frac{1}{4}(\beta_{22} - 2\beta_{12} + \beta_{11}) \leq 1.$$

For each set of  $\beta_{12}$  and  $\beta_{13}$ , Eq. (10) determines the unique value of  $\Delta$ , except for the intersecting point of the two regions (i) and (ii) for which  $\beta_1$  is an arbitrary. Thus, the  $\lambda$  surface in Eq. (10) contains two nontrivial  $\lambda$  lines of critical points given by

- (a)  $\beta_{11} = \beta_{13} = 2 \ln(2 + \sqrt{3})$ ,  $\beta_{12} = \beta_{22} = 0$ ,  $\beta_1$ -arbitrary,
- (b)  $\beta_{11} = \beta_{12} = \beta_{13} = 0$ ,  $\beta_{22} = 2 \ln(2 + \sqrt{3})$ ,  $\beta_1$ -arbitrary.

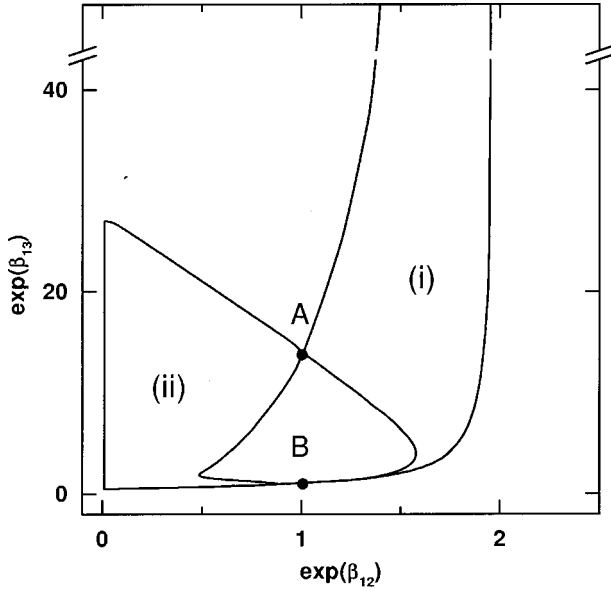


FIG. 1. The area in the plane of the coupling parameters  $\exp \beta_{12}$  and  $\exp \beta_{13}$  where the critical  $\lambda$  surface exists, i.e., conditions (9) and (i) or (ii) are satisfied. Points A and B correspond to projections of two nontrivial  $\lambda$  lines of (a) critical points and (b) for which  $\beta_1$  is arbitrary.

As shown in [18] the model exhibits a critical behavior different from the critical behavior elsewhere on the  $\lambda$  surface. The phase transition is not marked with the logarithmic divergence of the derivative of the order parameter  $P$ , where  $P$  is the quadrupolar moment defined as

$$P = \frac{1}{N} \sum_i^N \langle S_i^2 \rangle = Z^{-1} \sum_{\{S\}} S_i^2 \exp(-\beta H).$$

This phase transition is associated with the logarithmic divergence in the specific heat.

The area in the plane of coupling parameters  $\exp(\beta_{12})$  and  $\exp(\beta_{13})$ , where the  $\lambda$  surface exists is shown in Fig. 1. In the Appendix it is proved that this area is connected and there is no phase transition for  $T \rightarrow \infty$  and  $T \rightarrow 0$ . Hence, for all possible values of the coupling parameters there is only one finite critical value of the external field for which the phase transition is of the second-order, except points (a) and (b) in the above equation.

The spin- $\frac{3}{2}$  Ising model with Z(2) symmetry was investigated on a square lattice by Izmailian [17]. It was shown that this model is reducible to an eight-vertex model on a surface in the parameter space spanned by the coupling constants  $J$ ,  $K$ ,  $L$ , and  $M$ . It was also shown that this model is equivalent to an exactly solvable free fermion model along two lines in the parameter space. The two  $\lambda$  lines of a second-order phase transition was found exactly in this model.

In terms of our gauge theory, these  $\lambda$  lines have the following form:

(c)  $\beta_{11} = \beta_{12} = \beta_{13} = 0$ ,  $\beta_{22} = 2 \ln(1 + \sqrt{2})$ ,  $\beta_1$ -arbitrary,

(d)  $\beta_{11} = \beta_{13} = 2 \ln(1 + \sqrt{2})$ ,  $\beta_{12} = \beta_{22} = 0$ ,  $\beta_1$ -arbitrary.

On these  $\lambda$  lines our gauge theory exhibits an Ising-type second-order phase transition (logarithmic specific-heat singularity).

Thus, we showed that there exists an area (lines for square lattice) where the 2D generalized Z(4) Potts gauge model mapped to the corresponding 2D spin- $\frac{3}{2}$  Ising model. Hence, because of universality of critical indexes it follows that these two models have the same critical indexes and belong to the same universality class.

### III. CONCLUDING REMARKS

In summary, we have found an exact analytical solution of the Z(4) gauge-lattice model with a single and double plaquette representation of the action by mapping it to the dual spin- $\frac{3}{2}$  Ising model with Z(2) symmetry. For the model on the square lattice we found the  $\lambda$  lines of the second-order phase transition with logarithmic specific-heat singularity. For the model on the triangular lattice we derived the two  $\lambda$  surfaces of a second-order phase transition with a usual Ising-type singularity of the order parameter and two nontrivial  $\lambda$  lines of critical points on which our model exhibits the critical behavior unlike critical behavior elsewhere on the  $\lambda$  surfaces. We demonstrated that the 2D general Potts gauge model belongs to the same universality class as the 2D spin- $\frac{3}{2}$  Ising model.

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### APPENDIX

Here we present the analytic investigation of the  $\lambda$  surface in the plane of the coupling parameters  $\exp(\beta_{12})$  and  $\exp(\beta_{13})$  for triangular lattice. Let us make the following denotations  $x = \exp(\beta_{12})$ ,  $y_{11} = \exp(\beta_{13})$ , and  $z = \exp(\beta_{22})$ . After elimination of  $\beta_{11}$  from Eq. (9) one obtains the first-order polynomial for  $z$ ,

$$z(1 - 2y + xy) + (2 - x)(y - x) = 0. \quad (\text{A1})$$

In terms of the variables  $x$ ,  $y$ , and  $z$  the conditions (i) and (ii) take the following form:

$$(\text{Ai}) \quad 1 \leq zy(2 - x)/x^2 \leq c^2,$$

$$\text{and} \quad zy/(2 - x) \geq c^2,$$

$$(\text{Aii}) \quad 1 \leq zy/(2 - x) \geq c^2$$

$$\text{and} \quad zy(2 - x)/x^2 \geq c^2,$$

where  $c = 2 + \sqrt{3}$ . From (Ai) and (Aii) it is easy to show that  $x < 2$  for any values of  $z$  and  $y$ . Taking into account that for  $T \rightarrow 0$ ,  $x, y, z$  can take only values  $\{0, 1, \infty\}$  and for  $T \rightarrow \infty$   $x = y = z = 1$ , one can show that neither A(i) nor A(ii) are satisfied, hence there is no phase transition for  $T \rightarrow 0$  and  $T \rightarrow \infty$ . Using this fact one can construct the area in the plane of coupling parameters  $\exp(\beta_{12})$  and  $\exp(\beta_{13})$  in which Eq. (A1) and one of the conditions (Ai) or (Aii) are satisfied. This area is shown in Fig. 1.

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